DFPD-A-03-32

# Enhancement of Non-Gaussianity after Inflation

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## Abstract

We study the evolution of cosmological perturbations on large scales, up to second order, for a perfect fluid with generic equation of state. Taking advantage of super-horizon conservation laws, it is possible to follow the evolution of the non-Gaussianity of perturbations through the different stages after inflation. We find that a large non-linearity is generated by the gravitational dynamics from the original inflationary quantum fluctuations. This leads to a significant enhancement of the tiny intrinsic non-Gaussianity produced during inflation in single-field slow-roll models.

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# 1 Introduction

Inflation is the simplest and most successful mechanism proposed to date for the causal generation of primordial cosmological perturbations on cosmologically relevant scales [1]. The gravitational amplification of the primordial perturbations is supposed to seed structure formation in the Universe and produce Cosmic Microwave Background (CMB) anisotropies in agreement with observational data [2]. Due to the smallness of the primordial cosmological perturbations, their generation and evolution have usually been studied within linear theory [3]. After the seminal work by Tomita [4], only recently second-order perturbation theory [5–7] has been employed to evaluate specific physical observables generated during inflation [8,9]. The importance of an accurate determination of higher-order statistics as the bispectrum comes from the fact that they allow to search for the signature of non-Gaussianity in the primordial perturbations which is usually parametrized by a dimensionless non-linear parameter $f_{\rm NL}$. Indeed, a number of present and future CMB experiments, such as *WMAP* [10] and *Planck*, have enough resolution to either constrain or detect non-Gaussianity of CMB anisotropy data with high precision [11].

The main result of the second-order analysis performed in [8, 9] is that single-field slow-roll models of inflation give rise to a level of intrinsic non-Gaussianity which – at the end of the inflationary stage – is tiny, being first-order in the slow-roll parameters.

The goal of this paper is to study the post-inflationary evolution on super-horizon scales of the primordial non-linearity in the cosmological perturbations. We perform a fully relativistic analysis of the dynamics of second-order perturbations for a perfect fluid with generic equation of state taking advantage of the super-horizon conservation of the second-order gauge-invariant curvature perturbation recently discussed in Refs. [12, 13] (see also [8, 14, 15]). Our main result is that the post-inflationary evolution gives rise to an enhancement of the level of non-Gaussianity on super-horizon scales. Once again, inflation provides the key generating mechanism to produce super-horizon seeds, which are later amplified by gravity.

The plan of the paper is as follows. In Section 2 we provide the second-order expansion of the metric and of the energy-momentum tensor, assuming that the source term is represented by a perfect fluid with constant equation of state. In Section 3 we solve the perturbed Einstein equations up to first order around a Friedmann-Robertson-Walker background. The body of the paper is contained in Section 4, where we derive the super-horizon evolution equations of the second-order gravitational potential and density perturbations. Section 5 contains a brief discussion of our findings.

# 2 Perturbations of a flat Robertson-Walker Universe up to second order

In order to study the perturbed Einstein equations, we first write down the perturbations on a spatially flat Robertson-Walker background following the formalism of Refs. [5, 6].

We shall first consider the fluctuations of the metric, and then the fluctuations of the energy-momentum tensor. Hereafter greek indices run from 0 to 3, while latin indices label the spatial coordinates from 1 to 3. If not otherwise specified we will work with conformal time $\tau$, and a prime will stand for a derivative with respect to $\tau$.

## 2.1 The metric tensor

The components of a perturbed spatially flat Robertson-Walker metric can be written as

$$\begin{aligned} g_{00} &= -a^2(\tau)\left(1+2\phi^{(1)}+\phi^{(2)}\right) , \\ g_{0i} &= a^2(\tau)\left(\hat{\omega}_i^{(1)}+\frac{1}{2}\hat{\omega}_i^{(2)}\right) , \\ g_{ij} &= a^2(\tau)\left[(1-2\psi^{(1)}-\psi^{(2)})\delta_{ij}+\left(\hat{\chi}_{ij}^{(1)}+\frac{1}{2}\hat{\chi}_{ij}^{(2)}\right)\right] , \end{aligned} \tag{2.1}$$

where the scale factor $a$ is a function of the conformal time $\tau$. The standard splitting of the perturbations into scalar, transverse (*i.e* divergence-free) vector parts, and transverse trace-free tensor parts with respect to the 3-dimensional space with metric $\delta_{ij}$ can be performed in the following way:

$$\hat{\omega}_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)} , \tag{2.2}$$

$$\hat{\chi}_{ij}^{(r)} = D_{ij}\chi^{(r)} + \partial_i \chi_j^{(r)} + \partial_j \chi_i^{(r)} + \chi_{ij}^{(r)} , \tag{2.3}$$

where $(r) = (1), (2)$ stand for the order of the perturbations, $\omega_i$ and $\chi_i$ are transverse vectors ($\partial^i \omega_i^{(r)} = \partial^i \chi_i^{(r)} = 0$), $\chi_{ij}^{(r)}$ is a symmetric transverse and trace-free tensor ($\partial^i \chi_{ij}^{(r)} = 0$, $\chi^{i(r)}_{\ i} = 0$) and $D_{ij} = \partial_i \partial_j - (1/3)\, \delta_{ij}\, \partial^k \partial_k$ is a trace-free operator. Here and in the following latin indices are raised and lowered using $\delta^{ij}$ and $\delta_{ij}$, respectively.
For our purposes the metric in Eq. (2.1) can be simplified. In fact, first-order vector perturbations are zero; moreover, the tensor part gives a negligible contribution to second-order perturbations. Thus, in the following we can neglect $\omega_i^{(1)}$, $\chi_{(1)i}^{(r)}$ and $\chi_{(1)ij}^{(r)}$. However the same is not true for the second order perturbations. In the second-order theory the second-order vector and tensor contributions can be generated by the first-order scalar perturbations even if they are initially zero [6]. Thus we have to take them into account and we shall use the metric

$$\begin{aligned} g_{00} &= -a^2(\tau)\left(1+2\phi^{(1)}+\phi^{(2)}\right) , \\ g_{0i} &= a^2(\tau)\left(\partial_i\omega^{(1)}+\frac{1}{2}\,\partial_i\omega^{(2)}+\frac{1}{2}\,\omega_i^{(2)}\right) , \\ g_{ij} &= a^2(\tau)\left[\left(1-2\psi^{(1)}-\psi^{(2)}\right)\delta_{ij}+D_{ij}\left(\chi^{(1)}+\frac{1}{2}\chi^{(2)}\right)\right. \\ &+ \left.\frac{1}{2}\left(\partial_i\chi_j^{(2)}+\partial_j\chi_i^{(2)}+\chi_{ij}^{(2)}\right)\right] . \end{aligned} \tag{2.4}$$

The controvariant metric tensor is obtained by requiring (up to second order) that $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu{}^\lambda$ and it is given by

$$
\begin{aligned}
g^{00} &= -a^{-2}(\tau)\left(1-2\phi^{(1)}-\phi^{(2)}+4\left(\phi^{(1)}\right)^2-\partial^i\omega^{(1)}\partial_i\omega^{(1)}\right)\,,\\
g^{0i} &= a^{-2}(\tau)\left[\partial^i\omega^{(1)}+\frac{1}{2}\left(\partial^i\omega^{(2)}+\omega^{i(2)}\right)+2\left(\psi^{(1)}-\phi^{(1)}\right)\partial^i\omega^{(1)}-\partial^i\omega^{(1)}D^i{}_k\chi^{(1)}\right]\,,\\
g^{ij} &= a^{-2}(\tau)\left[\left(1+2\psi^{(1)}+\psi^{(2)}+4\left(\psi^{(1)}\right)^2\right)\delta^{ij}-D^{ij}\left(\chi^{(1)}+\frac{1}{2}\chi^{(2)}\right)\right.\\
&- \frac{1}{2}\left(\partial^i\chi^{j(2)}+\partial^j\chi^{i(2)}+\chi^{ij(2)}\right)-\partial^i\omega^{(1)}\partial^j\omega^{(1)}\\
&- \left.4\psi^{(1)}D^{ij}\chi^{(1)}+D^{ik}\chi^{(1)}D^j{}_k\chi^{(1)}\right]\,.
\end{aligned}
\tag{2.5}
$$

Using $g_{\mu\nu}$ and $g^{\mu\nu}$ one can calculate the connection coefficients and the Einstein tensor components up to second order in the metric fluctuations. They are given in the Appendix A of Ref. [8]. From now one, we will adopt the *Poisson gauge* [16] which is defined by the condition $\omega = \chi = \chi_i = 0$. Then, one scalar degree of freedom is eliminated from $g_{0i}$ and one scalar and two vector degrees of freedom from $g_{ij}$. This gauge generalizes the so-called longitudinal gauge to include vector and tensor modes and contains a solenoidal vector $\omega_i^{(2)}$.

## 2.2 Energy-momentum tensor of the fluid

Since after inflation and reheating the Universe enters a radiation-dominated phase and, subsequently, a matter- and dark energy-dominated phases, we shall consider a generic fluid characterized by an energy density $\rho$ and pressure $P$ with energy-momentum tensor

$$
T^\mu{}_\nu = (\rho+P)\,u^\mu u_\nu + P\delta^\mu{}_\nu\,, \tag{2.6}
$$

where $u^\mu$ is the four-velocity vector subject to the constraint $g^{\mu\nu}u_\mu u_\nu = -1$. At second order of perturbation theory it can be decomposed as

$$
u^\mu = \frac{1}{a}\left(\delta^\mu_0 + v^\mu_{(1)} + \frac{1}{2}v^\mu_{(2)}\right)\,. \tag{2.7}
$$

For the first- and second-order perturbations, we get

$$
\begin{aligned}
v^0_{(1)} &= -\psi^{(1)},\\
v^0_{(2)} &= -\phi^{(2)}+3\left(\psi^{(1)}\right)^2+v_i^{(1)}v^i_{(1)}\,.
\end{aligned}
\tag{2.8}
$$

Similarly, we obtain

$$
\begin{aligned}
u_0 &= a\left(-1-\phi^{(1)}-\frac{1}{2}\phi^{(2)}+\frac{1}{2}\left(\psi^{(1)}\right)^2-\frac{1}{2}v_i^{(1)}v^i_{(1)}\right)\,,\\
u_i &= a\left(v_i^{(1)}+\frac{1}{2}v_i^{(2)}-2\psi^{(1)}v_i^{(1)}+\frac{1}{2}\omega_i^{(2)}\right)\,.
\end{aligned}
\tag{2.9}
$$

The energy density $\rho$ can be split into a homogeneous background $\rho_0(\tau)$ and a perturbation $\delta\rho(\tau, x^i)$ as follows

$$\rho(\tau, x^i) = \rho_0(\tau) + \delta\rho(\tau, x^i) = \rho_0(\tau) + \delta^{(1)}\rho(\tau, x^i) + \frac{1}{2}\delta^{(2)}\rho(\tau, x^i)\,, \tag{2.10}$$

where the perturbation has been expanded into a first and a second-order part, respectively. The same decomposition can be adopted for the pressure $P$.

Using the expression (2.10) into Eq. (2.6) and calculating $T^\mu_{\ \nu}$ up to second order we find

$$T^\mu_{\ \nu} = T^{\mu(0)}_{\ \nu} + \delta^{(1)}T^\mu_{\ \nu} + \delta^{(2)}T^\mu_{\ \nu}\,, \tag{2.11}$$

where $T^{\mu(0)}_{\ \nu}$ corresponds to the background value, and

$$T^{0(0)}_{\ 0} + \delta^{(1)}T^0_{\ 0} = -\rho_0 - \delta^{(1)}\rho\,, \tag{2.12}$$

$$\delta^{(2)}T^0_{\ 0} = -\frac{1}{2}\delta^{(2)}\rho - (1+w)\,\rho_0 v_i^{(1)} v^i_{(1)}\,, \tag{2.13}$$

$$T^{i(0)}_{\ 0} + \delta^{(1)}T^i_{\ 0} = -\,(1+w)\,\rho_0 v^i_{(1)}\,, \tag{2.14}$$

$$\delta^{(2)}T^i_{\ 0} = -\,(1+w)\,\rho_0 \left[\left(\psi^{(1)} + \frac{\delta^{(1)}\rho}{\rho_0}\right) v^i_{(1)} + \frac{1}{2}v^i_{(2)}\right]\,, \tag{2.15}$$

$$T^{i(0)}_{\ j} + \delta^{(1)}T^i_{\ j} = w\rho_0\left(1 + \frac{\delta^{(1)}\rho}{\rho_0}\right)\delta^i_{\ j}\,, \tag{2.16}$$

$$\delta^{(2)}T^i_{\ j} = (1+w)\,\rho_0 v^i_{(1)} v_j^{(1)} + \frac{1}{2}w\delta^{(2)}\rho\delta^i_{\ j}\,. \tag{2.17}$$

In the previous expressions we have made the assumption that the pressure $P$ can be expressed in terms of the energy density as $P = w\rho$ with constant $w$.

# 3 Basic first-order Einstein equations on large-scales

Our starting point are the perturbed Einstein equations $\delta G^\mu_{\ \nu} = \kappa^2\,\delta T^\mu_{\ \nu}$ in the Poisson gauge. Here $\kappa^2 \equiv 8\pi\,G_{\rm N}$. At first-order, the $(0-0)$- and $(i-0)$-components of Einstein equations are

$$\frac{1}{a^2}\left[6\,\mathcal{H}^2\phi^{(1)} + 6\,\mathcal{H}\psi^{(1)\prime} - 2\nabla^2\psi^{(1)}\right] = -\kappa^2\delta^{(1)}\rho\,, \tag{3.1}$$

$$\frac{2}{a^2}\left(\mathcal{H}\partial^i\phi^{(1)} + \partial^i\psi^{(1)\prime}\right) = -\kappa^2\,(1+w)\,\rho_0 v^i_{(1)}\,, \tag{3.2}$$

where we have indicated by $\mathcal{H} = \frac{a'}{a}$ the Hubble rate in conformal time. These equations, together with the non-diagonal part of the $(i-j)$-component of Einstein equations, give $\psi^{(1)} = \phi^{(1)}$ and, on super-horizon scales,

$$\psi^{(1)} = -\frac{1}{2}\frac{\delta^{(1)}\rho}{\rho_0} = \frac{3(1+w)}{2}\mathcal{H}\frac{\delta^{(1)}\rho}{\rho_0'} \,. \tag{3.3}$$

The continuity equation yields an evolution equation for the large-scale energy density perturbation

$$\delta^{(1)}\rho' + 3\mathcal{H}\left(1+w\right)\delta^{(1)}\rho - 3\psi^{(1)'}\left(1+w\right)\rho_0 = \frac{2}{3}\frac{\rho_0}{\mathcal{H}^2}\nabla^2\left(\psi^{(1)'} + \mathcal{H}\psi^{(1)}\right) \,. \tag{3.4}$$

This equation, together with the the background continuity equation $\rho_0' + 3\mathcal{H}\left(1+w\right)\rho_0 = 0$, leads to the conservation on large-scales of the first-order gauge-invariant curvature perturbation [3]

$$\zeta^{(1)} = -\psi^{(1)} - \mathcal{H}\frac{\delta^{(1)}\rho}{\rho_0'} \,. \tag{3.5}$$

Indeed, both the density perturbation, $\delta\rho$ and the curvature perturbation, $\psi$, are in general gauge-dependent. Specifically, they depend upon the chosen time-slicing in an inhomogeneous universe. The curvature perturbation on fixed time hypersurfaces is a gauge-dependent quantity: after an arbitrary linear coordinate transformation at first-order, $t \to t + \delta t$, it transforms as $\psi^{(1)} \to \psi^{(1)} + \mathcal{H}\delta t$. For a scalar quantity, such as the energy density, the corresponding transformation is $\delta\rho^{(1)} \to \delta\rho^{(1)} - \rho_0'\delta t$. However the gauge-invariant combination $\zeta^{(1)}$ can be constructed which describes the density perturbation on uniform curvature slices or, equivalently the curvature of uniform density hypersurfaces. On large scales $\zeta^{(1)'} \simeq 0$. Using Eq. (3.3) and the background continuity equation, we can determine

$$\psi^{(1)} = -\frac{3(1+w)}{5+3w}\,\zeta^{(1)} \,, \tag{3.6}$$

which is useful to relate the curvature $\psi^{(1)}$ during either the matter or the radiation epoch to the gauge-invariant curvature perturbation $\zeta^{(1)}$ at the end of the inflationary stage. Indeed, since $\zeta^{(1)}$ is constant, we can write

$$\psi^{(1)} = -\frac{3(1+w)}{5+3w}\,\zeta_I^{(1)} \,, \tag{3.7}$$

where the subscript "$I$" means that $\zeta^{(1)}$ is computed during the inflationary stage.

# 4 Basic second-order Einstein equations on large-scales and non-Gaussianity

In order to determine the non-Gaussianity of the cosmological perturbations after inflation, we have to derive the behaviour on large-scales of the metric and the energy density perturbations at second order. Again, our starting point are the Einstein equations perturbed at second order $\delta^{(2)}G^{\mu}{}_{\nu} = \kappa^2\,\delta^{(2)}T^{\mu}{}_{\nu}$ in the Poisson gauge. The second-order expression for the Einstein tensor $\delta^{(2)}G^{\mu}{}_{\nu}$ can be found in any gauge in the Appendix A of Ref. [8] and we do not report it here.

- The $(0-0)$-component of Einstein equations (see Eq. (A.39) in Ref. [8]) leads to

$$
\begin{aligned}
&3\mathcal{H}^2\phi^{(2)} + 3\mathcal{H}\psi^{(2)\prime} - \nabla^2\psi^{(2)} - 12\mathcal{H}^2\left(\psi^{(1)}\right)^2 - 3\left(\nabla\psi^{(1)}\right)^2 \\
&-8\psi^{(1)}\nabla^2\psi^{(1)} - 3\left(\psi^{(1)\prime}\right)^2 = \kappa^2 a^2 \delta^{(2)} T^{0}_{\ 0}\,.
\end{aligned}
\tag{4.1}
$$

- A relation between the gravitational potentials at second-order $\psi^{(2)}$ and $\phi^{(2)}$ can be obtained from the traceless part of the ($i$-$j$) components of Einstain's equations (see Eqs. (A.42) and (A.43) in Ref. [8]). We find

$$
\begin{aligned}
\psi^{(2)} - \phi^{(2)} &= -4\left(\psi^{(1)}\right)^2 - \nabla^{-2}\left(2\partial^i\psi^{(1)}\partial_i\psi^{(1)} + 3\left(1+w\right)\mathcal{H}^2 v^i_{(1)} v_{(1)i}\right) \\
&+ 3\nabla^{-4}\partial_i\partial^j\left(2\partial^i\psi^{(1)}\partial_j\psi^{(1)} + 3\left(1+w\right)\mathcal{H}^2 v^i_{(1)} v_{(1)j}\right)\,.
\end{aligned}
\tag{4.2}
$$

  This constraint is the second-order equivalent of the linear constraint $\psi^{(1)} = \phi^{(1)}$ in the Poisson gauge.

- In order to close the system and fully determine the variables $\psi^{(2)}$, $\phi^{(2)}$ and $\delta^{(2)}\rho$, we use the energy conservation at second-order and the divergence of the $(i-0)$-component of Einstein equations (see Eq. (A.40) in Ref. [8])[4]

$$
\begin{aligned}
\delta^{(2)}\rho' &+ 3\mathcal{H}\left(1+w\right)\delta^{(2)}\rho - 3\left(1+w\right)\rho_0\psi^{(2)'} - 6(1+w)\psi^{(1)'}\left[\delta^{(1)}\rho + 2\rho_0\psi^{(1)}\right] \\
&= -2(1+w)\rho_0\left(v^{(1)}_i v^i_{(1)}\right)' - 2(1+w)(1-3w)\mathcal{H}\rho_0 v^{(1)}_i v^i_{(1)} \\
&+ 4(1+w)\rho_0\partial_i\psi^{(1)} v^i_{(1)} + 2\frac{\rho_0}{\mathcal{H}^2}\left(\psi^{(1)}\nabla^2\psi^{(1)\prime} - \psi^{(1)\prime}\nabla^2\psi^{(1)}\right)\,.
\end{aligned}
\tag{4.3}
$$

[4] Notice that Eq. (4.3) generalizes Eq. (5.33) of Ref. [13] and corrects a sign misprint in front of the fourth term of that equation.

This equation can be rewritten in a more suitable form

$$\begin{aligned}
&\left[\psi^{(2)}+\mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'}+(1+3w)\mathcal{H}^2\left(\frac{\delta\rho^{(1)}}{\rho_0'}\right)^2-4\mathcal{H}\left(\frac{\delta\rho^{(1)}}{\rho_0'}\right)\psi^{(1)}\right]'=\frac{2}{3}\left(v_i^{(1)}v^i_{(1)}\right)'\\
&+\frac{2}{3}(1-3w)\mathcal{H}v_i^{(1)}v^i_{(1)}-\frac{4}{3}\partial_i\psi^{(1)}v^i_{(1)}+\frac{16}{27\left(1+w\right)^2\mathcal{H}}\psi^{(1)}\nabla^2\psi^{(1)}\\
&-\frac{2}{3\left(1+w\right)\mathcal{H}^2}\left[\left(1-\frac{8}{9\left(1+w\right)}\right)\psi^{(1)}\nabla^2\psi^{(1)\prime}-\left(1-\frac{4\left(1+3w\right)}{9\left(1+w\right)}\right)\psi^{(1)\prime}\nabla^2\psi^{(1)}\right]\\
&+\frac{8\left(1+3w\right)}{27\left(1+w\right)^2\mathcal{H}^3}\left[\frac{\left(\nabla^2\psi^{(1)}\right)^2}{3}-\psi^{(1)\prime}\nabla^2\psi^{(1)\prime}+\frac{\nabla^2\psi^{(1)\prime}\nabla^2\psi^{(1)}}{3\mathcal{H}}\right], \qquad (4.4)
\end{aligned}$$

where the argument on the left-hand side can be further simplified to

$$\begin{aligned}
&\psi^{(2)}+\mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'}-\left(5+3w\right)\mathcal{H}^2\left(\frac{\delta^{(1)}\rho}{\rho_0'}\right)^2\\
&=\psi^{(2)}+\mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'}-\frac{4}{5+3w}\left(\zeta_I^{(1)}\right)^2 \qquad (4.5)
\end{aligned}$$

and the final form has been obtained employing Eq. (3.7) [17]. From Eqs. (4.4) and (4.5) we find

$$\psi^{(2)}+\mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'}-\left(5+3w\right)\mathcal{H}^2\left(\frac{\delta^{(1)}\rho}{\rho_0'}\right)^2=\mathcal{C}+\frac{2}{3}\left(v_i^{(1)}v^i_{(1)}\right)+\int^{\tau}d\tau'\,\mathcal{S}(\tau')\,, \qquad (4.6)$$

where $\mathcal{C}$ is a constant in time, $\mathcal{C}'=0$, on large-scales, and

$$\begin{aligned}
\mathcal{S} \;=\; &\frac{2}{3}(1-3w)\mathcal{H}v_i^{(1)}v^i_{(1)}-\frac{4}{3}\partial_i\psi^{(1)}v^i_{(1)}+\frac{16}{27\left(1+w\right)^2\mathcal{H}}\psi^{(1)}\nabla^2\psi^{(1)}\\
-\;&\frac{2}{3\left(1+w\right)\mathcal{H}^2}\left[\left(1-\frac{8}{9\left(1+w\right)}\right)\psi^{(1)}\nabla^2\psi^{(1)\prime}-\left(1-\frac{4\left(1+3w\right)}{9\left(1+w\right)}\right)\psi^{(1)\prime}\nabla^2\psi^{(1)}\right]\\
+\;&\frac{8\left(1+3w\right)}{27\left(1+w\right)^2\mathcal{H}^3}\left[\frac{\left(\nabla^2\psi^{(1)}\right)^2}{3}-\psi^{(1)\prime}\nabla^2\psi^{(1)\prime}+\frac{\nabla^2\psi^{(1)\prime}\nabla^2\psi^{(1)}}{3\mathcal{H}}\right]. \qquad (4.7)
\end{aligned}$$

## 4.1 Determination of the non-linearity parameter

Since we are interested in the determination of the non-linear parameter $f^{\phi}_{\rm NL}$ after the inflationary stage, it is convenient to fix the constant $\mathcal{C}$ by matching the conserved quantity at the end of inflation ($\tau=\tau_I$)

$$\mathcal{C}=\psi_I^{(2)}+\mathcal{H}_I\frac{\delta^{(2)}\rho_I}{\rho_{0I}'}-2\left(\zeta_I^{(1)}\right)^2, \qquad (4.8)$$

where we have used the fact that during inflation $w_I \simeq -1$.

The inflationary quantity $\left(\psi_I^{(2)} + \mathcal{H}_I \frac{\delta^{(2)}\rho_I}{\rho'_{0I}}\right)$ has been computed in Refs. [8,9]

$$\psi_I^{(2)} + \mathcal{H}_I \frac{\delta^{(2)}\rho_I}{\rho'_{0I}} \simeq (\eta - 3\epsilon)\left(\zeta_I^{(1)}\right)^2 + \mathcal{O}(\epsilon,\eta)\ (\mathrm{non-local\ terms})\,, \qquad (4.9)$$

in terms of the slow-roll parameters $\epsilon = 1 - \mathcal{H}'_I/\mathcal{H}_I^2$ and $\eta = 1 + \epsilon - (\varphi''/\mathcal{H}_I\varphi')$ where $\mathcal{H}_I$ is the Hubble parameter during inflation and $\varphi$ is the inflaton field driving the exponential growth of the scale factor during inflation [1]. Since during inflation the slow-roll parameters are tiny, we can safely disregard the intrinsically second-order terms originated from the inflationary epoch.

Combining Eqs. (4.1), (4.2), (4.6) and (4.8) we single out an equation for the gravitational potential $\phi^{(2)}$ on large scales

$$\begin{aligned}
\phi^{(2)\prime} + \frac{5+3w}{2}\mathcal{H}\phi^{(2)} &= (5+3w)\mathcal{H}\left(\psi^{(1)}\right)^2 + \frac{3}{2}\mathcal{H}(1+w)\left[\nabla^{-2}\left(2\partial^i\psi^{(1)}\partial_i\psi^{(1)}\right.\right.\\
&+ \left. 3\,(1+w)\,\mathcal{H}^2 v^i_{(1)}v_{(1)i}\right) - 3\nabla^{-4}\partial_i\partial^j\left(2\partial^i\psi^{(1)}\partial_j\psi^{(1)}\right.\\
&+ \left.\left. 3\ (1+w)\,\mathcal{H}^2 v^i_{(1)}v_{(1)j}\right)\right] + \frac{3}{2}\mathcal{H}(1+w)\int_{\tau_I}^{\tau}\mathcal{S}(\tau')d\tau' + \frac{1}{\mathcal{H}}\left(\nabla\psi^{(1)}\right)^2\\
&+ \frac{8}{3\mathcal{H}}\psi^{(1)}\left(\nabla^2\psi^{(1)}\right) + \frac{\nabla^2\mathcal{S}_1}{3\mathcal{H}} + \frac{1}{\mathcal{H}}\left(\psi^{(1)\prime}\right)^2 - \mathcal{S}'_1\,,
\end{aligned} \qquad (4.10)$$

where $\mathcal{S}_1$ denotes the R.H.S. of Eq (4.2).

We want to integrate this equation from $\tau_I$ to a time $\tau$ in the matter-dominated epoch. The general solution is given by the solution of the homogeneous equation plus a particular solution

$$\begin{aligned}
\phi^{(2)} &= \phi^{(2)}(\tau_I)\exp\left[-\int_{\tau_I}^{\tau}\frac{5+3w}{2}\mathcal{H}d\tau'\right]\\
&+ \exp\left[-\int_{\tau_I}^{\tau}\frac{5+3w}{2}\mathcal{H}d\tau'\right]\times\int_{\tau_I}^{\tau}\exp\left[\int_{\tau_I}^{\tau'}\frac{5+3w}{2}\mathcal{H}ds\right]b(\tau')d\tau'\,,
\end{aligned} \qquad (4.11)$$

where $b(\tau)$ stands for the source term in the R.H.S of Eq. (4.10). Notice that the homogeneous solution during both the radiation and the matter-dominated epoch decreases in time. Therefore we can neglect the homogeneous solution and focus on the contributions from the source term $b(\tau)$. At a time $\tau$ in the matter-dominated epoch $\exp[-\int_{\tau_I}^{\tau} d\tau'\,\mathcal{H}\,(5+3w)/2] \propto \tau^{-5}$. Thus if we are interested in the gravitational potential $\phi^{(2)}$ during the matter dominated epoch the contributions in the particular solution coming from the radiation-dominated epoch can be considered negligible. Recalling that during the matter-dominated epoch the linear gravitational potential $\psi^{(1)}$ is constant in

time, it turns out that

$$\begin{aligned}\phi^{(2)} &\simeq 2\left(\psi^{(1)}\right)^2 + \frac{3}{5}\left[\nabla^{-2}\left(\frac{10}{3}\partial^i\psi^{(1)}\partial_i\psi^{(1)}\right) - 3\nabla^{-4}\partial_i\partial^j\left(\frac{10}{3}\partial^i\psi^{(1)}\partial_j\psi^{(1)}\right)\right]\\ &+ \exp\left[-\int_{\tau_I}^{\tau}\frac{5+3w}{2}\mathcal{H}d\tau'\right]\times\int_{\tau_I}^{\tau}\exp\left[\int_{\tau_I}^{\tau'}\frac{5+3w}{2}\mathcal{H}ds\right]\left\{\frac{3}{2}\mathcal{H}(1+w)\int_{\tau_I}^{\tau'}\mathcal{S}(s)ds\right.\\ &+ \left.\frac{1}{\mathcal{H}}\left(\nabla\psi^{(1)}\right)^2 + \frac{8}{3\mathcal{H}}\psi^{(1)}\left(\nabla^2\psi^{(1)}\right) + \frac{\nabla^2\mathcal{S}_1}{3\mathcal{H}}\right\}d\tau'\,, \end{aligned} \tag{4.12}$$

where we have used Eq. (3.2) to express the first-order velocities in terms of the gravitational potential, and we have taken into account that during the matter-dominated epoch $\mathcal{S}_1' = 0$.

As the gravitational potential $\psi^{(1)}$ on super-horizon scales is generated during inflation, it is clear that the origin of the non-linearity traces back to the inflationary quantum fluctuations.

The total curvature perturbation will then have a non-Gaussian ($\chi^2$)-component. For instance, the lapse function $\phi = \phi^{(1)} + \frac{1}{2}\phi^{(2)}$ can be expressed in momentum space as

$$\phi(\mathbf{k}) = \phi^{(1)}(\mathbf{k}) + \frac{1}{(2\pi)^3}\int d^3k_1\, d^3k_2\, \delta^{(3)}\left(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}\right)\, f_{\rm NL}^{\phi}\left(\mathbf{k}_1,\mathbf{k}_2\right)\phi^{(1)}(\mathbf{k}_1)\phi^{(1)}(\mathbf{k}_2)\,, \tag{4.13}$$

where we have defined an effective "momentum-dependent" non-linearity parameter $f_{\rm NL}^{\phi}$. Here the linear lapse function $\phi^{(1)} = \psi^{(1)}$ is a Gaussian random field. The gravitational potential bispectrum reads

$$\langle\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\rangle = (2\pi)^3\,\delta^{(3)}\left(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\right)\left[2\, f_{\rm NL}^{\phi}\left(\mathbf{k}_1,\mathbf{k}_2\right)\mathcal{P}_\phi(k_1)\mathcal{P}_\phi(k_2) + \text{cyclic}\right]\,, \tag{4.14}$$

where $\mathcal{P}_\phi(k)$ is the power-spectrum of the gravitational potential.

At this point, in order to give the non-linearity parameter, an important remark is in order. Indeed, when dealing with second-order perturbations which are expressed in terms of first-order quantities, also the short-wavelength behaviour of the first-order perturbations must be taken into account, as it becomes evident when going to momentum space. The crucial point here is which is the final quantity one is interested in. We are interested in calculating the bispectrum of the gravitational potential on large scales as a measure of non-Gaussianity of the cosmological perturbations on those scales. The bispectrum of such quantities is twice the kernel which appears when expressing the second-order quantities in terms of first-order ones in Fourier space, an example of such a kernel being $f_{\rm NL}^{\phi}\left(\mathbf{k}_1,\mathbf{k}_2\right)$ in Eq. (4.13). This means that, when calculating the bispectrum, we can evaluate the kernel in the long-wavelength limit, irrespective of the integration over the whole range of momenta. This is the reason why the last term in Eq. (4.12) gives a negligible contribution to the large-scale limit of the gravitational potential bispectrum.

Therefore, going to momentum space, from Eq. (4.12) we directly read the corresponding non-linearity parameter for scales entering the horizon during the matter-dominated stage

$$f^{\phi}_{\rm NL} \simeq -\frac{1}{2} + 4\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k^2} - 3\frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k^4} + \frac{3}{2}\frac{k_1^4 + k_2^4}{k^4} \tag{4.15}$$

where $k = |\mathbf{k}_1 + \mathbf{k}_2|$.

The non-Gaussianity provided by expression (4.15) will add to the known Newtonian and relativistic second-order contributions which are relevant on sub-horizon scales, such as the Rees-Sciama effect [18], whose detailed analysis has been given in Refs. [19].

# 5 Conclusions

In this paper we have provided a framework to study the evolution of non-linearities present in the primordial cosmological perturbations seeded by inflation on super-horizon scales. The tiny non-Gaussianity generated during the inflationary epoch driven by a single scalar field gets enhanced in the post-inflationary stages giving rise to a non-negligible signature of non-linearity in the gravitational potentials. On the other hand, there are many physically motivated inflationary models which can easily accomodate for a primordial value of $f_{\rm NL}$ larger than unity. This is the case, for instance, of a large class of multi-field inflation models which leads to either non-Gaussian isocurvature perturbations [20] or cross-correlated non-Gaussian adiabatic and isocurvature modes [21]. Other interesting possibilities include the "curvaton" model, where the late time decay of a scalar field other than the inflaton induces curvature perturbations [22], and the so-called "inhomogeneous reheating" mechanism where the curvature perturbations are generated by spatial variations of the inflaton decay rate [23]. Our findings indicate that a positive future detection of non-linearity in the CMB anisotropy pattern will not rule out single field models as responsible for seeding structure formation in our Universe.


## Acknowledgements

It is a pleasure to thank David Wands for several discussions. We also thank the referee for her/his comments which allowed to strengthen the presentation of our results and to correct for an algebraic error in the final formula.


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$$
\begin{aligned}
\zeta^{(2)} &= -\psi^{(2)} - \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'} + 2\mathcal{H}\frac{\delta^{(1)}\rho'}{\rho_0'}\frac{\delta^{(1)}\rho}{\rho_0'} \\
&+ 2\frac{\delta^{(1)}\rho}{\rho_0'}\left(\psi^{(1)\prime} + 2\mathcal{H}\psi^{(1)}\right) - \left(\frac{\delta^{(1)}\rho}{\rho_0'}\right)^2\left(\mathcal{H}\frac{\rho_0''}{\rho_0'} - \mathcal{H}' - 2\mathcal{H}^2\right) \\
&= -\psi^{(2)} - \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'} - (1+3w)\mathcal{H}^2\left(\frac{\delta\rho^{(1)}}{\rho_0'}\right)^2 + 4\mathcal{H}\left(\frac{\delta\rho^{(1)}}{\rho_0'}\right)\psi^{(1)}\,,
\end{aligned}
$$

where in the last passage we have made use of the first-order continuity equation (3.4). The quantity $\zeta^{(2)}$ satisfies the conservation equation $\zeta^{(2)\prime} = 4\zeta^{(1)}\zeta^{(1)\prime}$ leading to

$\zeta^{(2)} - 2\left(\zeta^{(1)}\right)^2 =$ constant. Since $\zeta^{(1)}$ is conserved on large-scales, this implies that $\zeta^{(2)}$ is conserved as well. Incidentally, we note that the combination $\zeta^{(2)} - 2\left(\zeta^{(1)}\right)^2$ is equal to the conserved quantity defined in Ref. [12]. Of course, every quantity differing from $\zeta^{(2)}$ by $c\left(\zeta^{(1)}\right)^2$ with $c$ an arbitrary constant, is constant on super-horizon scales.

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